Robust multiattribute decision making under risk and uncertainty in engineering design

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In this article, the problem of choosing from a set of design alternatives based upon multiple, conflicting, and uncertain criteria is investigated. The problem of selection over multiple attributes becomes harder when risky alternatives exist. The overlap measure method developed in this article models two sources of uncertainities – imprecise or risky attribute values provided to the decision maker and inabilities of the decision-maker to specify an exact desirable attribute value. Effects of these uncertainties are mitigated using the overlap measure metric. A subroutine to this method, called the robust alternative selection method, ensures that the winning alternative is insensitive to changes in the relative importance of the different design attributes. The overlap measure method can be used to model and handle various sources of uncertainties and can be applied to any number of multiattribute decision-making methods. In this article, it is applied to the hypothetical equivalents and inequivalents method, which is a multiattribute selection method under certainty.

Keywords: Multiattribute decision making; Robust solution; Uncertainty; Risk

1. Introduction

All products and processes used in our daily lives are the results of a series of decisions, many of which are made by engineers. Decision-based design (DBD) takes the perspective that an engineering design and optimization process can be viewed as a set of decisions which can be modeled and solved in order to provide effective decision support for engineers (Shupe 1988). The fundamental phases of DBD are to determine all possible design alternatives and to choose the best one (Hazelrigg 1998, Chen et al. 2000, Chen and Wassenaar 2003). There exist many methods rooted in innovative thinking and creative problem solving, which can be used to determine possible design alternatives. Non-traditional approaches to alternative generation include grammar-based approaches (Li et al. 2004), experimental design techniques (Phadke 1989, Montgomery 1997), and evolutionary algorithms (Zechman and Ranjithan 2004). Although the phase of DBD that deals with generating design alternatives is an extensive area of research in itself, it is the second phase that is the primary focus of this article. This article deals with the decision-making aspect of DBD, once some information of the possible design alternatives is obtained.

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Selection of an optimal design from a set of alternatives can be based on either a single attribute or multiple, independent attributes with decisions made using complete information (certainty) or under uncertainty and risk. Classification of decision problems is shown in Table 1 and discussed subsequently.

For problems that are based on a single attribute under certainty, a simple rank ordering of the alternatives will provide the best solution. For single attribute problems under uncertainty, single attribute utility theory (SAUT) is used to determine the alternative of choice (von Neumann and Morgenstern 1953, Keeney and Raiffa 1993). However, decisions are rarely made based on a single attribute or criteria but rather engineering design decisions are made based on a very large number of attributes.

Many methods have been developed in the past and are available in the literature for multi-attribute decision-making under certainty and uncertainty. Some of these methods are briefly reviewed subsequently and their validity is discussed by Hazelrigg (2003). The hypothetical equivalents and inequivalents method (HEIM), which is a method for multiattribute decision making under certainty and the basis for the work in this article, is also a part of this review. Detailed reviews of the following methods are given by See et al. (2004), where the method details as well as the advantages and pitfalls are presented. In this article, the methods for certainty are briefly mentioned for the sake of completeness.

The pairwise comparisons method compares two design alternatives at a time and forms the basis of the analytical hierarchy process (AHP) (Basak and Saaty 1993, Saari 2000). Its pitfalls include design cycling and negligence of strength of preferences and relative attribute importances. When alternatives are rank ordered and then given scores with respect to each attribute, pitfalls include violating the independence of irrelevant alternatives (IIA) principle (Peter and Wakker 1991) and ignoring attribute importance and strength of preferences. Attribute normalization along with non-linear strength of preference functions can be used to satisfy the IIA principle. In addition, weights can be assigned to each attribute and a score can be determined for each alternative by evaluating the aggregation of the normalized attribute ratings. The primary disadvantage is that a precise technique to determine the attribute weights does not exist.

The HEIM has been developed to avoid the pitfalls of many of these methods, most notably providing a sound approach to determining accurate attribute weights (Gurnani et al. 2003). The advantage of using hypothetical alternatives to elicit decision maker preferences is that the decision maker is not biased towards any one of the actual alternatives and can state his or her true preferences. Once the preferences over hypothetical alternatives have been stated, an optimization problem is formulated, where the attribute weights are set as design variables, preference statements are set as constraints, and a pseudo-objective function (to drive the sum of the weights to one) is used. The standard optimization problem is shown in equation (1).

\[
\text{Minimize } F(x) = \left(1 - \sum_{j=1}^{n} w_j \right)^2
\]

Subject to \( h(x) = 0 \) \( g(x) \leq 0 \) (1)

<table>
<thead>
<tr>
<th>Single attribute under certainty</th>
<th>Multiple attributes under certainty</th>
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<tbody>
<tr>
<td>\textit{Solution method}: optimization</td>
<td>\textit{Solution method}: Ranking, HEIM, etc.</td>
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</table>

<table>
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<tr>
<th>Single attribute under uncertainty</th>
<th>Multiple attributes under uncertainty</th>
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<tr>
<td>\textit{Solution method}: single attribute utility</td>
<td>\textit{Solution method}: Focus of paper</td>
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</table>

Table 1. Classification of selection decision problems.
where, \( h(x), g(x) \) are the constraints based on the decision maker’s preferences towards the hypothetical alternatives and \( x \) is the vector of attribute weights. To illustrate how the constraints are constructed, consider two hypothetical alternatives \( H_1 \) and \( H_2 \). The overall values of the hypothetical alternatives \( H_1 \) and \( H_2 \) are the summations of the attribute weights multiplied by the attribute rating for each hypothetical alternative. Mathematically, this is shown in equations (2) and (3),

\[
H_1 = \sum_{i=1}^{n} w_i r_i
\]

\[
H_2 = \sum_{i=1}^{n} w_i r'_i
\]

where, \( w_i \) is the weight of the \( i \)th attribute (the attribute weights are chosen as design variables), \( r_i \) is the rating of the \( i \)th attribute for hypothetical alternative \( H_1 \), and \( r'_i \) is the rating of the \( i \)th attribute for hypothetical alternative \( H_2 \).

On the basis of the stated preference over the hypothetical alternatives, for example, \( H_1 > H_2 \), \( H_1 < H_2 \), or \( H_1 = H_2 \), an inequality or equality equation is determined using equations (2) and (3). The symbol ‘\( > \)’ implies ‘is preferred more than’ and ‘\( < \)’ implies ‘is preferred less than’. This forms the constraint \( h(x) \) (for equality preferences) or \( g(x) \) (for inequality preferences) in the optimization problem of equation (1). Note that an additive model is shown in equation (1). However, HEIM can be used with any attribute aggregation scheme and is not bound by the summation methodology. The authors seek to clearly distinguish between the weighted sum method (arbitrary assignment of attribute weights) and HEIM (optimization problem solution for attribute weights), to prevent confusion later in the article where attribute ratings are multiplied with weight ‘variables’ and summed over all attributes to determine a total score for each alternative. It is important to note that the total score is a function of the weight ‘variables’.

Most of the decisions in engineering design are made using imperfect models, imprecise information, and limited knowledge. Therefore, most of these decisions are made under uncertainty, which is the focus of this article. As an approach to handling these types of decisions, multiattribute utility theory (MAUT) was introduced by Keeney and Raiffa (1993) and used in engineering design by Thurston (1991) and Li and Azarm (2000). MAUT presents a method of attribute information aggregation and uses attribute weights to achieve this. However, the biggest drawback of MAUT is that the attribute weight assignment is arbitrary. Additionally, it is tedious and time consuming to construct the multiattribute utility function (Thurston et al. 1994).

The evidential reasoning method developed by Yang and Sen (1994a,b) for multiattribute decision making under uncertainty requires attributes to be broken down into objective and subjective factors. The factors can be closely related to each other, and it is required that the assessment of the factors is consistent, which is hard to do. The use of fuzzy sets is one method to determine an assessment of the various factors (Wang et al. 1994) that are combined using the Dempster–Shafer theory (Rogova et al. 1998). Some of the pitfalls of this method are that it is a very tedious algorithm where assessments of factors are combined at one level, which are again combined on the basis of the different scenarios being considered, making the process highly complex. Additionally, it is highly sensitive to the grade scheme being used for the assessment of the subjective factors.

The next two methods for multiattribute decision making lay some of the foundation of the method developed in this article.

Suh (1995) developed the probability of success metric to assess the amount of overlap between a design’s feasible range and a customer’s desired range over an attribute. The higher
the overlap, the greater is the probability of the design to be successful and hence should be the alternative of choice. The second axiom of Suh’s axiomatic design states that the best design is the one that has the minimum information content, where information is a function of the range of commonality between the designer and customer’s preferred ranges. In addition, the customer-based expected utility metric, as developed by Besharati et al. (2002), combines probabilities associated with risky outcomes, possible market share that can be obtained, and the decision maker’s utility functions. The primary drawback of this metric is that for multiple attributes, the use of a multiattribute utility function is proposed, the limitations of which have been already discussed.

A pitfall of these two metrics is that the relevant attributes used to assess the engineering aspect of a product might not be the same attributes used by customers to assess a product. Although this work does not overcome this pitfall, it is developed with the assumption that the decision is being made within one realm. That is, the attributes used to assess the alternatives are relevant to the engineering discipline, and the decision maker is the design engineer, not the customer. The methods developed in this article can be used in different disciplines such as marketing or engineering, as long as the attributes assessing the alternatives are also the attributes relevant to the decision maker.

In this article, a single decision maker has to choose from among a set of discrete design alternatives on the basis of multiple attributes. The alternatives are known to be risky because each alternative has a different outcome with corresponding probabilities (Keeney and Raiffa 1993). Additionally, uncertainty in the problem exists due to the risky attribute values and multiattribute nature of the problem, which requires the decision maker to make tradeoffs among various attributes. Owing to this, the decision maker is unable to clearly state his or her preferred value for a single attribute and hence specifies a preference structure (utility function) over each attribute individually. Using the overlap measure method developed in section 3, these sources of uncertainty are handled such that their effects are mitigated. To illustrate the developed methodology, the overlap measure method is applied to the selection of an airplane from a choice of different airplanes, where the attribute values are risky.

2. Problem formulation

Consider the case of a single decision maker selecting a design from a set of alternatives \(a_i, i = 1, 2, \ldots, m\), on the basis of different attributes \(x_j, j = 1, 2, \ldots, n\). In addition to the attributes and alternatives, the decision maker has a utility function associated with each of the attributes \(x_j\), denoted as \(U_j(x)\). This utility function models the uncertainties associated with the decision maker’s inability to clearly state his or her preferred attribute level. In addition, there exists uncertainty associated with the performance of each alternative with respect to each attribute. The fundamental challenge is to appropriately model these uncertainties and convert them to a cardinal score in order to make an informed and accurate decision, where the alternative with the best total score is the alternative of choice. However, fundamental challenges exist in determining the total score, namely,

(i) converting the attribute probability distribution into a single, meaningful, dimensionless score,
(ii) incorporating the decision maker’s preference structure, and
(iii) determining the value of the weights (getting a robust solution).

In section 3, the overlap measure method that addresses each of these challenges is presented. As an illustration, the overlap measure method is applied to the selection of a single airplane
from a set of four possible alternatives on the basis of three attributes. The details of this example problem are provided subsequently.

Consider a fictional airline carrier, Jetair, who is planning to establish an air fleet to serve the routes between major cites in the USA. It is assumed that Jetair has decided to purchase only one type of aircraft for its entire fleet to reduce operating costs, similar to the strategy used by Southwest Airlines and Jetblue Airways. At this point, Jetair has identified four possible choices that meet its requirements and budget constraints. These choices are simplistically called Airplane 1, Airplane 2, Airplane 3, and Airplane 4. The attributes of interest to Jetair are $x_1$ the cruise speed, $x_2$ the flight range, and $x_3$ the passenger capacity.

The aforementioned attributes are assumed uncertain. Thus, $x_1$, $x_2$, and $x_3$ are defined as random variables with associated probability distributions. In this article, it is assumed that $x_1$ has a normally distributed probability density function (PDF), that is, for each airplane, there is a mean cruise speed and an associated variance. The random variable $x_2$ is assumed to have a uniform PDF, whereas $x_3$ is assumed to be a discrete distributed random variable. The parameter values of the three random variables for each of the four alternatives are given in Table 2.

In Table 2, LB and UB refer to the lower and upper bounds of the uniformly distributed random variable $x_2$. For example, Airplane 1 can have a cruise range between 8500 and 9200 nmi.

In addition to the risky nature of the attributes, it is also assumed that the decision maker is unable to specify his or her preference for the attributes with certainty and has a utility function associated with the preference structure. It is assumed that the decision maker has a risk neutral utility function for the attribute cruise speed, a risk averse utility function for cruise range, and a uniform utility function for passenger capacity. The utility functions that model the decision maker’s preferences are presented next.

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**Table 2. Attribute values for alternatives of airplane selection problem.**

<table>
<thead>
<tr>
<th>Type of aircraft</th>
<th>Cruise speed (Mach)</th>
<th>Cruise range (nmi)</th>
<th>Passenger capacity</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Variance</td>
<td>LB</td>
</tr>
<tr>
<td>Aircraft 1</td>
<td>0.84</td>
<td>0.02</td>
<td>8500</td>
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<tr>
<td>Aircraft 2</td>
<td>0.85</td>
<td>0.02</td>
<td>6600</td>
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<tr>
<td>Aircraft 3</td>
<td>0.85</td>
<td>0.02</td>
<td>6000</td>
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<tr>
<td>Aircraft 4</td>
<td>0.86</td>
<td>0.02</td>
<td>7700</td>
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<td></td>
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</table>
2.1 Cruise speed

To determine the utility function for cruise speed, the lowest and highest feasible speeds among all the alternatives are necessary. These speeds are shown in equation (4).

\[
\text{Lowest cruise speed} = \mu_i - 3\sigma_i \\
\text{Highest cruise speed} = \mu_i + 3\sigma_i \tag{4}
\]

where \( \mu_i \) represents the mean cruise speed for alternative \( i \) and \( \sigma_i \) represents the variance of alternative \( i \). As all the alternatives have the same variance, and Aircraft 1 has the lowest mean speed, the lowest speed value is \( \mu_{11} - 3\sigma_{11} = 0.84 - (3 \times 0.02) = 0.78 \) Mach, and by definition, the utility of this speed is zero, \( U(0.78) = 0 \). Similarly, Aircraft 4 has the highest speed, the value of which is \( \mu_{41} + 3\sigma_{41} = 0.86 + (3 \times 0.02) = 0.92 \) Mach, and by definition, the utility of this speed is one, \( U(0.92) = 1 \).

As the decision maker’s preference is assumed risk neutral, a straight line is fitted through the two end points and the utility function is given in equation (5).

\[
U_1(x) = 7.1429x - 5.5714 \tag{5}
\]

2.2 Cruise range

The lowest cruise range value over all alternatives is 6000 nmi and the highest value is 9200 nmi. These two points correspond to utilities of zero and one, respectively. The risk attitude of the decision maker is assumed risk averse. The generic form of a risk averse utility function is

\[
U(x) = a - b e^{cx} \tag{6}
\]

The values of the constants are determined using a three-point fit. As a risk averse attitude is assumed, the midway utility point (\( U = 0.5 \)) is assumed to correspond to a cruise range value of 7000 nmi. Using a least squares minimization to perform the three-point fit, the utility function for cruise range is obtained as shown in equation (7).

\[
U_2(x) = 1.2342 - 27.8519 e^{-0.00052x} \tag{7}
\]

2.3 Passenger capacity

The decision maker prefers that the airplane hold between 300 and 350 passengers. Thus, the utility function for the passenger capacity is given in equation (8).

\[
U_3(x) = \begin{cases} 
0 & \text{if } x < 300 \text{ or } x > 350 \\
1 & \text{if } 300 \leq x \leq 350 
\end{cases} \tag{8}
\]

Given the alternatives of choice and the corresponding attribute utility functions, the overlap measure method is presented in section 3.

3. The overlap measure method

The flowchart of the overlap measure method is shown in figure 1. The various boxes in the flowchart are explained step by step and applied to the aircraft selection problem simultaneously.
Step 1. Generate hypothetical alternatives – An experimental design (Phadke 1989) is used to generate the hypothetical alternatives. The hypothetical alternatives are represented as uncertain probability distributions and as this article deals with risky alternatives, the hypothetical alternatives of the example problem are also defined using PDFs. The hypothetical alternatives for the airplane selection problem are given in table 3.

Step 2. Determine the overlap measure metric – the overlap measure is a metric that combines the uncertain range of the attribute value for a given alternative and the decision maker’s preference function for that attribute, to determine a dimensionless score. A broader discussion of the overlap measure as well as the application of Steps 2–4 for the airplane selection problem is provided in section 4.

Table 3. Hypothetical alternatives for airplane selection problem.

<table>
<thead>
<tr>
<th>Type of aircraft</th>
<th>Speed (Mach)</th>
<th>Range (nmi)</th>
<th>Number of passengers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Variance</td>
<td>LB</td>
</tr>
<tr>
<td>$H_1$</td>
<td>0.84</td>
<td>0.02</td>
<td>6000</td>
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</tr>
<tr>
<td>$H_2$</td>
<td>0.86</td>
<td>0.02</td>
<td>6000</td>
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<tr>
<td>$H_3$</td>
<td>0.84</td>
<td>0.02</td>
<td>8500</td>
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<tr>
<td>$H_4$</td>
<td>0.86</td>
<td>0.02</td>
<td>6600</td>
</tr>
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</table>
Step 3. Formulate the preference structure to form constraints and solve – On the basis of the preference statements made by the decision maker over the hypothetical alternatives developed in Step 3, constraints are setup as specified in the HEIM formulation. These are then combined to determine the attribute weights by solving the optimization problem of equation (1).

Step 4. Check for robustness – by the choice of the objective function in equation (1), it is possible that multiple sets of feasible attribute weights can be found that result in different winning alternatives. The robust alternative selection method (RASM) is used to determine one robust winning alternative. Section 5 addresses this step with a detailed discussion of the RASM.

Now that the overlap measure method framework has been presented, a discussion of the overlap measure metric is given section 4.

4. The overlap measure metric

Let \( f_{ij}(x) \) represent the PDF of the \( j \)th attribute, for some alternative \( i \). The decision maker’s utility function for the \( j \)th attribute is denoted as \( U_j(x) \). For each alternative \( i \), the \( j \)th attribute score, defined as the overlap measure, is evaluated by:

\[
\text{Overlap measure } O_{ij} = \int_{-\infty}^{+\infty} f_{ij}(x) U_j(x) \, dx
\]

where \( O_{ij} \) is the overlap measure score of the \( i \)th alternative for the \( j \)th attribute, \( f_{ij}(x) \) the PDF of the \( i \)th alternative for the \( j \)th attribute, \( U_j(x) \) the utility function of the \( j \)th attribute, and \( x \) the integrating variable denoting the attribute.

Graphically, the scenario is shown in figure 2. The integral limits in equation (9) go from \(-\infty\) to \(+\infty\), indicating that the integral is evaluated over the entire set of feasible values for that attribute, assuming that the attribute is defined to be a continuous random variable. However, in figure 2, the double vertical lines encompass the actual region-of-interest. The integrating region might reduce in cases where either the utility function or the PDF is zero or the PDF has insignificant probability values outside this region. For example, we have assumed that for a normal PDF, the region-of-interest lies \( 3\sigma \) on either side of the mean. Therefore, the limits of integration reduce from \((-\infty, +\infty)\) to \((\mu - 3\sigma, \mu + 3\sigma)\).

![Figure 2. Graphical representation of overlap measure for the \( i \)th alternative over the \( j \)th attribute.](image-url)
For the case where the attribute is a discrete random variable (for example, passenger capacity), the PDF becomes a probability mass function and the integral \( \int \) is replaced with a summation \( \sum \) for the overlap measure calculation.

The overlap measure is the integration of the product of a PDF and a utility function. A PDF always results in a non-negative number and utility functions are defined such that they also result in non-negative values. Hence, the lowest value for both the PDF and the utility function is 0, whereas the highest value for both the functions is 1. Therefore, the range of any overlap measure value is between 0 and 1,

\[
0 \leq O_{ij} \leq 1
\]

where an overlap measure of 0 implies that the alternative’s design range does not satisfy the utility function for an attribute, whereas a value of 1 implies that the design range completely satisfies the decision maker’s preferences. Values in between imply partial satisfaction.

Note that depending on the weights of the attributes, alternatives with high overlap measure values have a higher possibility of being the alternative of choice. In addition, the overlap measure is a dimensionless quantity, as shown in equation (10), facilitating its use when making comparisons. As both probability and utilities are dimensionless quantities, the overlap measure is dimensionless.

Overlap measure

\[
O_{ij} = \int_{-\infty}^{+\infty} f_{ij}(x)U_j(x) \, dx
\]

\[
= \int_{-\infty}^{+\infty} \frac{\text{probability}}{\text{units of attribute } j} \times \text{utilities} \times \text{units of attribute } j
\]

\[
= \text{utilities} \int_{-\infty}^{+\infty} \text{probability}
\]

Once the overlap measure is determined for all the attributes for each alternative, the total score function is determined by multiplying the overlap measure value with the attribute weight variables and summed over all attributes. Thus, the overlap measure index solves the first two challenges listed in section 2. That is, it converts a distribution into a single, meaningful score, and incorporates the decision maker’s preference structure. Once the score is determined, the problem reduces to a standard multiattribute selection problem under certainty, which is then solved using HEIM.

As an illustration, the overlap measure metric is evaluated for the airplane selection problem using the generic overlap measure equation (equation (9)). For example, the overlap measure metric is evaluated for Airplane 1 over the cruise speed attribute. The PDF of the cruise speed is given in equation (11).

\[
f_{11}(x) = \frac{1}{\sqrt{2\pi} \sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} = \frac{1}{\sqrt{2\pi} \times 0.02} e^{-(x-0.84)^2/0.02} \quad (11)
\]

Substituting equations (5) and (11) into equation (9), the overlap measure metric is calculated in equation (12).

\[
O_{11} = \int_{\mu_1-3\sigma_1}^{\mu_1+3\sigma_1} f_{11}(x)U_1(x) \, dx
\]

\[
= \int_{0.78}^{0.9} \frac{1}{\sqrt{2\pi} \times 0.02} e^{-(x-0.84)^2/2 \times (0.02)^2} (7.1429x - 5.5714) \, dx = 0.4437 \quad (12)
\]

The overlap measure values and the total score functions for the actual and hypothetical alternatives are provided in table 4.
The next step of the overlap measure method flowchart is to formulate the decision maker’s preferences into constraints. The constraints form part of the optimization problem of HEIM (equation (1)). For this example, the preference structure as shown in equation (13), is assumed:

\[ H_4 > H_2 \quad H_3 > H_2 \quad H_3 > H_1 \]  

(13)

Thus, the constraints are formulated using the total score functions of table 4 and the preference structure of equation (13). The inequality constraints are shown in equations (14)–(16).

\[ H_4 > H_2 : \quad 0.5865w_1 + 0.4977w_2 + 0.53w_3 > 0.5865w_1 + 0.2752w_2 + w_3 \]
\[ - 0.2225w_2 + 0.47w_3 + \delta \leq 0 \]  

(14)

\[ H_3 > H_2 : \quad 0.4437w_1 + 0.9533w_2 + 0.5w_3 > 0.5865w_1 + 0.2752w_2 + w_3 \]
\[ 0.1428w_1 - 0.6781w_2 + 0.5w_3 + \delta \leq 0 \]  

(15)

\[ H_3 > H_1 : \quad 0.4437w_1 + 0.9533w_2 + 0.5w_3 > 0.4437w_1 + 0.2752w_2 + 0.36w_3 \]
\[ - 0.6781w_2 - 0.14w_3 + \delta \leq 0 \]  

(16)

Note that the constraints are strictly ‘less than’ inequalities, and to convert them to ‘less than or equal to’, a small number, \( \delta \), is added to the left hand side of the equations. The optimization problem in standard form is given in equation (17).

\[
\begin{align*}
\text{Minimize} & \quad F(\mathbf{w}) = \left(1 - \sum_{j=1}^{3} w_j\right)^2 \\
\text{Subject to} & \quad -0.2225w_2 + 0.47w_3 + \delta \leq 0 \\
& \quad 0.1428w_1 - 0.6781w_2 + 0.5w_3 + \delta \leq 0 \\
& \quad -0.6781w_2 - 0.14w_3 + \delta \leq 0 \\
& \quad 0 \leq w_1, w_2, w_3 \leq 1
\end{align*}
\]

(17)

where \( \delta = 0.001 \).

A single solution to the optimization problem of equation (17) is:

\[
\begin{align*}
w_1 &= 0.402 \\
w_2 &= 0.531 \\
w_3 &= 0.067
\end{align*}
\]

(18)
The values for the weights, as calculated using HEIM, indicate that the most preferred attribute is cruise range, with cruise speed coming second. The weight value for passenger capacity is quite low as compared to the other two attributes, but is not small enough to ignore. The corresponding total scores for each alternative are calculated from the functions shown in Table 4. The weights shown in equation (18) are substituted into the functions and the evaluated values are shown subsequently.

\[
\text{Aircraft 1: } 0.4437w_1 + 0.9533w_2 + 0.3w_3 = 0.70467 \\
\text{Aircraft 2: } 0.5151w_1 + 0.4977w_2 + 0.5w_3 = 0.50485 \\
\text{Aircraft 3: } 0.5151w_1 + 0.2752w_2 + 0.16w_3 = 0.36392 \\
\text{Aircraft 4: } 0.5865w_1 + 0.7923w_2 + 0.8w_3 = 0.71008
\]

Thus, the winning alternative is Aircraft 4. This aircraft has the highest attribute rating for cruise speed, second highest rating for cruise range, and the highest rating for passenger capacity.

As the weights are found by using their sum as the objective function, theoretically there could exist different sets of weights that sum to one that satisfy all constraints and result in a different winning alternative. Using a different starting point to solve the optimization problem results in the following different set of feasible weight values that sum to as:

\[
\begin{align*}
w_1 &= 0.3 \\
w_2 &= 0.6 \\
w_3 &= 0.1
\end{align*}
\]

The winning alternative with these values is Aircraft 1. Comparing equations (18) and (19), there is a relatively small change in the magnitude of the attribute weight values, and the relative ordering of the weights does not change. However, even such a small change in the weights has a significant effect on the winning alternative because it changes from Aircraft 4 to Aircraft 1. Thus, the solution is not robust because the optimization problem used to find the attribute weights is under-constrained. To further constrain the feasible space, one option would be to use more hypothetical alternatives and determine the decision maker’s preferences over these initially. However, it is not possible to know a priori how many more constraints would be required to sufficiently constrain the feasible space in order to obtain one winning alternative. To overcome this and identify a single robust solution, the RASM is developed. This forms the last step in the overlap measure method flowchart and is discussed in section 5.

5. Robust alternative selection method

The RASM has been developed to ensure that there is only one winning alternative. The classical definition of robust is ‘a solution that is insensitive to variations in control and noise factors’ (Phadke 1989). In the context of this work, robust implies that the winning alternative is insensitive to changes in the values of the attribute weights.

RASM is developed to handle any number of attributes for a multiattribute selection problem. However, as the example problem has three attributes and hence three design variables (weights), a 3D axes system is used to visually represent the design space. A visual tool is developed using the OpenGL programming API (Neider et al. 1994) to better understand
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RASM. As the weight values cannot be negative, the region-of-interest is restricted to the first quadrant. Additionally, it is known that the attribute weights have to sum to one. That is,

\[ w_1 + w_2 + w_3 = 1 \]  \hspace{1cm} (20)

Figure 3 shows the visualized 3D world with the outline of the plane (represented by equation (20)) shown in black and the weights aligned along each axis.

The steps within the RASM are as follows.

Sub-step 1. Generate points (combinations of weights) in the feasible space, as defined by the preference constraints and the constraint that the sum of the weights must equal one. A grid or random point generation technique can be used. For the aircraft selection problem, 200 random numbers between 0 and 1 are generated for the weight values, and those points that satisfy all constraints are stored in a vector.

Sub-step 2. Evaluate the total score functions for all alternatives at each point and determine the winning alternative for that point. In general, \( k \) possible winning alternatives could exist, where \( k \leq m \). Figure 4 shows the weight values that satisfy all constraints specified in equation (17), with the points in black representing Aircraft 1 and those in gray representing Aircraft 4 as the winning alternative.

As seen in figure 4, Aircraft 4 is predominantly the winning alternative and there exists a small region of points that result in Aircraft 1 being the alternative of choice. Thus, for this problem, there are two possible winning alternatives from the original choice of four. Therefore, \( k = 2 \).

Sub-step 3. Where \( k > 1 \), equate the total score functions of all winning alternatives, taking two at a time. Thus, set

\[ V(a_l) = V(a_m) \hspace{1cm} l \neq m \hspace{1cm} l, m = 1, \ldots, k \]  \hspace{1cm} (21)

where \( a_l \) is the \( l \)th alternative and \( V(a_l) \) the total score of the \( l \)th alternative (last column of table 4).
The total scores of the winning alternatives are given as:

\[
V(a_l) = b_{l1}w_1 + b_{l2}w_2 + \cdots + b_{ln}w_n \\
V(a_m) = b_{m1}w_1 + b_{m2}w_2 + \cdots + b_{mn}w_n
\]  

(22)

where \(b_{l1}, b_{l2}, \ldots, b_{ln}\) are the \(n\) normalized attribute ratings for alternative \(a_l\) and \(w_1, w_2, \ldots, w_n\) the attribute weights.

Substituting equation (22) in equation (21),

\[
\begin{align*}
b_{l1}w_1 + b_{l2}w_2 + \cdots + b_{ln}w_n &= b_{m1}w_1 + b_{m2}w_2 + \cdots + b_{mn}w_n \\
(b_{l1} - b_{m1})w_1 + (b_{l2} - b_{m2})w_2 + \cdots + (b_{ln} - b_{mn})w_n &= 0 \\
d_1w_1 + d_2w_2 + \cdots + d_nw_n &= 0
\end{align*}
\]  

(23)

where

\[
d_i = b_{li} - b_{mi} \quad i = 1, n
\]

It is important to note that when \(n = 3\), equation (23) is the equation of a plane. The intersection of this plane with the \(\sum_{i=1}^{3} w_i = 1\) plane is the equation of the straight line that lies between the winning alternative regions. All sets of weight values that lie on this line will have the same total score value, implying that for these weight values, the decision maker is indifferent to the two winning alternatives. This is similar to the ‘indifference points’ concept, defined by Scott and Antonsson (2000), which seeks attribute values that are equally preferable to the designer. Within the realm of RASM, this set of indifference points lies on a straight line for a three attribute problem and on a hyperplane for a problem with more than three attributes.

For the example problem, the winning alternatives are Aircraft 1 and Aircraft 4. The corresponding total scores, therefore, are:

\[
\begin{align*}
V(\text{Aircraft 1}) &= 0.4437w_1 + 0.9533w_2 + 0.3w_3 \\
V(\text{Aircraft 4}) &= 0.5865w_1 + 0.7923w_2 + 0.8w_3
\end{align*}
\]
The two winning total scores are equated in equation (24).

\[
V(\text{Aircraft 1}) = V(\text{Aircraft 4})
\]

\[
0.4437w_1 + 0.9533w_2 + 0.3w_3 = 0.5865w_1 + 0.7923w_2 + 0.8w_3 \\
-0.1428w_1 + 0.161w_2 - 0.5w_3 = 0
\]  

(24)

**Sub-step 4.** Develop new hypothetical alternatives from equation (23). To accomplish this, the \(d_i\)'s are rearranged into two positive numbers such that two new hypothetical alternatives are generated. The rearrangement should be such that meaningful hypothetical alternatives are generated. This means that the normalized attribute ratings are such that some part of the attribute span is covered. It is noted that no rearrangement of equation (23) can lead to one hypothetical alternative completely dominating another. To see this, consider a two attribute problem with two winning alternatives \(A\) and \(B\) having the total score functions as shown in equation (25).

\[
V(\text{A}) = a_{A1}w_1 + a_{A2}w_2 \\
V(\text{B}) = a_{B1}w_1 + a_{B2}w_2
\]  

(25)

where \(a_{xy}\) is the rating of attribute \(y\) for alternative \(x\). For both the alternatives \(A\) and \(B\) to be robust optimal winning alternatives, their ratings will be ordered such that one alternative does not dominate another. Assume the following ordering for the rating of alternatives \(A\) and \(B\).

\[
a_{A1} < a_{B1} \\
a_{A2} > a_{B2}
\]  

(26)

Let \(H\) and \(G\) be two hypothetical alternatives that are formed from the rearrangement of alternatives \(A\) and \(B\). The total score functions of \(H\) and \(G\) are given in equation (27).

\[
V(\text{H}) = b_{H1}w_1 + b_{H2}w_2 \\
V(\text{G}) = b_{G1}w_1 + b_{G2}w_2
\]  

(27)

where \(b_{xy}\) is the rating of the hypothetical alternative \(x\) for attribute \(y\).

From equation (23), it is known that

\[
a_{A1} - a_{B1} = b_{H1} - b_{G1} \\
a_{A2} - a_{B2} = b_{H2} - b_{G2}
\]  

(28)

Using equations (26) and (28), it is seen that

\[
b_{H1} < b_{G1} \\
b_{H2} > b_{G2}
\]  

(29)

implying that hypothetical alternative \(H\) does not dominate hypothetical alternative \(G\). The structure of equation (26) can be extended to more attributes and implies that the attribute ratings have to be of conflicting nature for the value of the alternatives to be equal. Thus, the new hypothetical alternatives will also retain the same characteristics of conflicting attribute ratings.
Implementing Sub-step 4 in the example problem, the new hypothetical alternatives are derived from equation (24) and shown in equation (30).

\[-0.1428w_1 + 0.161w_2 - 0.5w_3 = 0\]

\[\implies 0.8572w_1 + w_2 + 0.5w_3 = w_1 + 0.839w_2 + w_3\]  

(30)

It is important to note that equation (30) is just one possible rearrangement of equation (24). Thus, the total score functions of the new hypothetical alternatives, denoted as $H_5$ and $H_6$, are:

$H_5 = 0.8572w_1 + w_2 + 0.5w_3$

$H_6 = w_1 + 0.839w_2 + w_3$  

(31)

Equation (31) represents the total score functions of two new hypothetical alternatives from which we seek to derive actual alternative values to be able to state preferences between $H_5$ and $H_6$. In table 4, the hypothetical alternatives had risky attribute values, similar to the actual alternatives of table 3. However, the central idea behind RASM is to strategically generate new hypothetical alternatives to capture designer preferences and determine the winning alternative. Additionally, because it is easier to state preferences over certain alternatives as opposed to uncertain ones, certainty equivalent attribute values for the two new hypothetical alternatives are determined using equations (4), (7) and (8). For example, hypothetical alternative $H_6$ has a coefficient of 1 for cruise speed in equation (31). This corresponds to a value of the cruise speed where the decision maker has a utility of 1. Using the decision maker’s utility function of equation (4) with a utility of 1 utile,

$U_1(x) = 7.1429x - 5.5714 = 1$

\[\implies x = 0.92 \text{ Mach}\]  

(32)

The actual alternative values for the newly generated hypothetical alternatives are shown in table 5.

Note that the attribute values for passenger capacity are still risky. This is purely due to the nature of the utility function, where a utility of 1 is obtained for a number of passengers between 300 and 350 and is 0 for all other values. To get overlap measure values of 0.5, as seen in equation (31), risky attribute values are needed.

Sub-step 5. Using the new hypothetical alternatives, the decision maker elicits his or her preferences. For each pairwise assessment of hypothetical alternatives, another constraint is added to the optimization problem. Sub-steps 3 through 5 are repeated for all winning alternatives to determine a robust feasible region where only one alternative is the winner.

Illustrating this step using the example problem, the decision maker specifies his or her preferences over alternatives $H_5$ and $H_6$. If the preference stated is $H_5 > H_6$, the

<table>
<thead>
<tr>
<th>Type of aircraft</th>
<th>Speed (Mach)</th>
<th>Range (nmi)</th>
<th>Number of passengers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_5$</td>
<td>0.9</td>
<td>9200</td>
<td>320</td>
</tr>
<tr>
<td></td>
<td></td>
<td>340</td>
<td></td>
</tr>
<tr>
<td>$H_6$</td>
<td>0.92</td>
<td>8184</td>
<td>330</td>
</tr>
</tbody>
</table>
Figure 5. Robust winning alternative aircraft 1.

Incorporating equation (33) within the feasible space of the 3D world shown in figure 4, a modified feasible space is obtained, as shown in figure 5. All feasible points are now colored black, implying that all possible solutions to the weights result in Aircraft 1 as being the winner.

In contrast, if the decision maker chooses \( H_6 \) over \( H_5 \),

\[
H_5 > H_6: \quad 0.8572w_1 + w_2 + 0.5w_3 > w_1 + 0.839w_2 + w_3 \\
0.1428w_1 - 0.161w_2 + 0.5w_3 + \delta \leq 0 \quad (33)
\]

Incorporating equation (34) in determining the set of feasible weights, figure 4 is modified to the 3D space shown in figure 6.

Figure 6. Robust winning alternative aircraft 4.
Now, all feasible weight values are colored gray, implying Aircraft 4 as the robust alternative of choice. As RASM applies to two winning alternatives at a time, and there are only two possible winners in the example problem, Steps 3 through 5 are not repeated. In problems with more winning alternatives, iteration is necessary.

Thus, it is concluded that RASM provides the necessary steps to eliminate the existence of multiple feasible solutions and ensures a single winning alternative for different feasible weight values.

6. Conclusions

In this article, the overlap measure method is presented for multiattribute selection problems under uncertainty. Some of the important conclusions are as follows.

(1) The overlap measure method is able to handle risky attribute values for different alternatives. This is important in engineering design because often exact outcomes of design alternatives are not known, but rather a set of outcomes with associated probabilities is the only information available when making selection decisions.

(2) The overlap measure method is able to mitigate the effects of the uncertainties that arise due to the risky nature of the attribute values.

(3) The overlap measure method incorporates the use of HEIM as a decision making method. HEIM is a decision making tool requiring the decision maker to make simplistic choices between pairs of alternatives in order to gather sufficient information to reach a decision.

(4) The overlap measure method ensures a robust winning alternative through the use of the RASM. RASM provides a systematic approach to determining the attribute weights such that the winning alternative is robust to small changes in the weight values.

Sources of future work include an investigation of robustness when the preference structure over the hypothetical alternatives (equation (13)) is altered. Additionally, the overlap measure method is developed for a single decision maker but in engineering design, rarely is a decision made by an individual and this method would need to be extended to group decision making. An important assumption in this work was that information regarding the risky attribute values for the different alternatives was available. A practical scenario that would need to be studied as a part of future work would be to develop the overlap measure method for truly uncertain problems, where the PDFs for the different attributes were unavailable.

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References


Yang, J. and Sen, P., Combining objective and subjective factors in multiple criteria marine design. In 5th International Marine Design Conference and Summer Meeting of German Society of Naval Architects, 1994b.