ENHANCED CONVERGENCE IN DISTRIBUTED DESIGN PROCESSES

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ABSTRACT

The focus of this paper is on studying the convergence properties of the solution process of decentralized or distributed sub-systems, where each subsystem has its own design problem, including objective(s), constraints, and design variables. The challenging aspect of this type of problem comes in the coupling of the subsystems, which create complex research and implementation challenges in modeling and solving these types of problems. We focus on the dynamics of these distributed design problems and attempt to further the understanding of the fundamental mechanics behind these processes in order to support the decisions being made by a network of decision makers. In this work, the domain of attraction, or region where convergence to a stable equilibrium point is guaranteed, of a decentralized design process is studied. Two approaches based on concepts from nonlinear control theory are presented: the first determines the domain of attraction for a specified Lyapunov function and the second optimizes for a Lyapunov function which maximizes the domain of attraction. The two techniques are illustrated on a benchmark pressure vessel design problem.

Keywords: Decentralized Design, Lyapunov Theory, Semidefinite Programming, Sum of Squares.

1 INTRODUCTION

The complexity of system design problems is one reason for the decomposition and decentralization of decisions. For example, The Economist reports that it took 700 parts to make the model Ford T, while modern cars pack many more in their radio alone [1]. From the aerospace industry, there are 3 million parts in a Boeing 777 provided by more than 900 suppliers [2]. Another reason for the decomposition of systems into smaller coupled subproblems is the multidisciplinary nature of these systems, which makes it impossible for one designer, or even a single design team, to consider the entire system as a single design problem. Typically, in complex systems, breaking it up into smaller units or subsystems will make the system more manageable [3, 4]. This decentralization of decisions is unavoidable in a large organization where having only one centralized decision maker is usually not applicable [5]. In fact, decentralization is recommended as a way to speed up product development processes and decrease the computational time and the complexity of the problem [6]. As an example, Airbus first decomposes along the main sections of the airplane, and is then further decomposed into smaller disciplinary subsystems as it is multidisciplinary along “Centres of Excellence” and “Centres of Competence” [7].

While the decomposition of complex problems certainly creates a series of smaller, less complex problems, it also creates several challenging issues associated with the coordination, convergence, and stability of these less complex problems. Previous work has been done on the coordination of decomposed problems using Design Structure Matrices [8], hierarchical approaches [9], or by effectively propagating the desirable top level design specifications to appropriate subsystems [10, 11]. The ef-
iciency of these approaches is compared in [12]. Another set of
techniques that have been used extensively to solve this kind
of problem include methods and constructs from game theory.
Previous work in Game Theory includes work to model the in-
teractions between the designers if several design variables are
shared among designers [13]. In [14], Game Theory is formally
presented as a method to help designers make strategic decisions
in a scientific way. In [15], distributed collaborative design is
viewed as a non-cooperative game, and maintenance considera-
tions are introduced into a design problem using concepts from
Game Theory. In [16], the manufacturability of multi-agent pro-
cess planning systems is studied using Game Theory concepts.
In [17], non-cooperative protocols are studied and the applica-
tion of Stackelberg leader/follower solutions is shown. Also
in [18], a Game Theory approach is used to address and describe
a multifunctional team approach for concurrent parametric de-
design. This set of work on game theory in complex systems de-
sign studied a set of valuable research issues, but did not for-
mally address the mechanisms of convergence in a generic de-
centralized design problem. Issues of convergence are critical to
study and resolve in the context of these kinds of systems design
problems [19], especially when decomposed problems are solved
asynchronously [20].

The issue of convergence (and divergence) was introduced
in [21], and then studied formally in [22] using geometric se-
ries theory for some basic decentralized problems marked by
quadratic and unconstrained problems. This work was then ex-
panded in [23] to nonlinear, constrained problems using concepts
from linear and nonlinear control theory. An important concept
that has emerged from these studies is the idea of a "domain
of attraction" of an equilibrium point, or a region where if the
distributed process starts, the entire process will converge to an
equilibrium solution point. These domains of attractions become
critical when the decomposed problems are nonlinear and there-
fore contain multiple equilibrium solution points with different
design and stability properties [23]. Further, for processes that
are marked by potentially divergent behavior, identifying a ro-
bust domain of attraction would be very attractive in systems co-
ordination and solution.

In [23], the domain of attraction was approximated by using
Lyapunov functions. A Lyapunov function, \( V \), is a positive
definite scalar function which in some cases can be thought of as
the energy of a given system (for physical systems). If the time
derivative of this function is negative definite in a domain \( D \), then
the system is said to be stable at the origin and \( D \) its domain of
attraction. This is explained in detail in section 3. In [23], the
equilibrium point of a given decentralized design problem is first
determined and then moved to the origin whose domain of attrac-
tion is estimated by determining the region where the Lyapunov
function constraints are satisfied. These domains, while identi-
fied for the first time in the context of systems design problems,
were limited in their range. Therefore, in this paper, we develop
new techniques similar to the one used by [24] to expand both
the science of modeling and identifying domains of attraction,
and the application to design problems marked by coupled de-
composition. Some of the operating assumptions of this work include:

1. The focus is on complex engineering systems, or those sys-
tems that necessitate the decomposition of the system into
smaller subsystems in order to reduce the complexity of the
design problems that must be solved.
2. The decomposed problems are "self-sufficient" in that they
contain their own set of independent design variables that
they control, their own design objective(s), and limitations
on the design in the form of constraints.
3. These decomposed problems can be represented in mathe-
matical formulations.
4. The coupling between the decomposed problems occurs
through the design variables. That is the values of the de-
sign variables of one decomposed problem are necessary in
at least one of the other decomposed problems.

In section 2, we present the basic background for the fund-
damental model we develop of decentralized design problems:
game theory and discrete time series. In section 3, the concept of
domain of attraction for a discrete time system is described
and Lyapunov-based techniques for the determination of the domain
of attraction are presented. Two approaches are presented: the
first determines the domain of attraction for a specified Lyapunov
function. The second solves for an optimal Lyapunov function
that maximizes the provable domain of attraction. In section 4,
the proposed techniques are illustrated on a benchmark pressure
vessel design problem. Section 5 includes some conclusions and
directions for future work.

2 BACKGROUND

Table 1 from [23] presents the Game-Theoretic formulation
for an optimization design problem with two subsystems where
\( x_1 \) and \( x_2 \) represent the vectors of design variables controlled by
subsystems 1 and 2 respectively. \( x_{1c} \) and \( x_{2c} \) are the non-local
design variables, or variables that appear in a model but are con-
trolled by the other subsystem. In a standard non-cooperative
protocol, players make decisions by assuming the choices of
other decision-makers. A solution \((x_{1N}, x_{2N})\) is a Nash solution if

\[
F_1(x_{1N}, x_{2N}) = \min_{x_1} F_1(x_1, x_{2N})
\]

and

\[
F_2(x_{1N}, x_{2N}) = \min_{x_2} F_2(x_{1N}, x_2)
\]

The Nash Equilibrium is the fixed point of two subsets of

\[ 2 \] Copyright © 2006 by ASME
Player 1’s Model: \( \text{Minimize} \quad F_1(x_1, x_{2c}) \)

subject to \( g^1_j(x_1, x_{2c}) \leq 0 \quad j = 1..m_1 \)

\( h^1_k(x_1, x_{2c}) = 0 \quad k = 1..I_1 \)

\( x_{1L} \leq x_1 \leq x_{1U} \)

Player 2’s Model: \( \text{Minimize} \quad F_2(x_2, x_{1c}) \)

subject to \( g^2_j(x_2, x_{1c}) \leq 0 \quad j = 1..m_2 \)

\( h^2_k(x_2, x_{1c}) = 0 \quad k = 1..I_2 \)

\( x_{2L} \leq x_2 \leq x_{2U} \)

Table 1. MULTI-PLAYER OPTIMIZATION PROBLEM FORMULATION

\( X \) the feasible space:

\[ (x_{1N}, x_{2N}) \in X_{1N}(x_{2N}) \times X_{2N}(x_{1N}) \quad (2) \]

where \[ X_{1N}(x_{2N}) = \{x_{1N} | F_1(x_{1N}, x_{2N}) = \min_{x_{1N}} F_1(x_1, x_2) \} \]

\[ X_{2N}(x_{1N}) = \{x_{2N} | F_2(x_{1N}, x_{2N}) = \min_{x_{2N}} F_2(x_1, x_2) \} \]

\( X_{1N}(x_{2N}) \) and \( X_{2N}(x_{1N}) \) are called the Rational Reaction Sets of the two subsystems.

Our basic model of a decentralized design problem is based on the assumption that the decision makers in the system make new decisions based on information from the previous iteration. At each iteration, each decision maker makes a decision that is consistent with their rational reaction set. That is, they return to their RRS, which minimizes their objective function. This process of iterating until convergence, while not antagonistic in nature as the term "noncooperative" sometimes implies, produces the equivalent of a noncooperative outcome. Indeed, in the design of a complex system it is desirable for the decision makers to cooperate when possible by sharing design model information, including objectives and constraints. However, this is rarely the case in a decentralized design process due to a number of types of barriers, including geographic constraints, asynchronous processes, complex analysis codes, differences in technology, confidentiality issues, and communication hurdles. Therefore, a decentralized process where decision makers iteratively share a series of final design solutions allows for the use of noncooperative game theory principles to guide its modeling and study. Convergence in this context means that this series of exchanging solutions eventually brings the decision makers to an acceptable solution that none have any incentive to change. This state of collective stability where no decision maker wants to change their solution is characteristic of the Nash equilibrium solution given in Eq. (2).

Irrespective of the number of decision makers, this series of iterative decisions among the decision makers can be modeled as a discrete time control problem as given in Eq. (3)

\[ x((k+1)) = f(x(k)) \quad (3) \]

In Eq. (3), as explained in [23], \( f \) is a representation of the RRS of each subsystem (exact or an approximation). This model dictates that the new values of the design variables, \( x \), are a function of the RRS representation of the information from the previous iteration. If the decentralized systems converges to a solution, then it occurs at the intersection of the subsystem RRS’s, which is an equilibrium point as shown in Eq. (2). There can be more than one equilibrium point or none depending on the problem. Given an equilibrium point, its domain of attraction provides valuable information. The domain of attraction of an equilibrium point, in the context of decentralized design, is the set of all points from which if the design process is initiated, the solution will converge to the equilibrium point. Thus obtaining a good estimate of the domain of attraction is very important. The following section describes methods to arrive at better estimates of domains of attraction.

3 ESTIMATING THE DOMAIN OF ATTRACTION

Consider the model of a dynamic system in discrete time given in Eq. (3) and assume that the origin is the point of equilibrium. The domain of attraction \( D \) is defined as:

\[ D = \{x(0) \in \mathbb{R}^n | x(t, x(0)) \rightarrow 0 \text{ as } t \rightarrow \infty \} \quad (4) \]

In other words, any state trajectory starting from within the domain of attraction will converge to the equilibrium point.

A Lyapunov function \( V \) is a scalar function which satisfies the constraints [25]:

\[ V(0) = 0, V(x(k)) > 0 \text{ for } D - \{0\} \quad (5) \]

\[ \Delta V = V(x(k+1)) - V(x(k)) \leq 0 \text{ in } D. \quad (6) \]

For any system, the existence of a Lyapunov function implies that the equilibrium point at the origin is stable. Further if

\[ \Delta V < 0 \text{ in } D, \quad (7) \]

the origin is asymptotically stable. If \( \Delta V < 0 \), within a Lyapunov surface (level set) \( V = \beta \), the state trajectory (the evolution of states with time) will pierce Lyapunov surfaces with smaller \( \beta \) over time, which implies that the states will never leave the region \( V \leq \beta \). It is clear that \( V < \beta \) represents a subset of the domain of attraction and maximizing \( \beta \) gives the optimal estimate of the domain of attraction for the Lyapunov function \( V \).
Seiler [26] proposed a technique using Sum of Squares (SOS) programming to determine the provable domain of attraction of the origin (equilibrium point) given a Lyapunov function for a continuous-time system. For any Lyapunov function \( V \), a region \( V \leq \beta \) is determined wherein \( V < 0 \) and hence belongs to the domain of attraction. The following method to determine provable domain of attraction was originally proposed by [26] for continuous time systems. In the next section, we present a modified version for discrete time applications.

### 3.1 Zero Detection Algorithm

Given a Lyapunov function \( V \), which shows that the origin is stable, consider the problem of estimating the domain of attraction of the origin. The problem is to maximize the region contained within \( V = \gamma \) where \( \Delta V \leq 0 \) and can be formulated as

\[
\text{max } \gamma \\
\text{subject to}
\Delta V = V(x(k+1)) - V(x(k)) \leq 0 \text{ when } V \leq \gamma \tag{8}
\]

Consider the implication

\[
\Delta V = 0 \Rightarrow x = 0 \text{ or } V \geq \gamma
\]

This implication verifies that \( \Delta V \neq 0 \) in the region \( V < \gamma \), except at the origin. Since \( \Delta V \) is a continuous function and the origin is stable, \( \Delta V < 0 \) in the region \( V < \gamma \). The implication is true if there exists a polynomial \( r(x) \) such that

\[
(V - \gamma)x^T x + r(x) \Delta V \geq 0. \tag{9}
\]

It can be seen from Eq. (9) that when \( \Delta V = 0, V \geq \gamma \) or \( x = 0 \) and hence constraining \( \Delta V \) to be negative within \( V = \gamma \) as explained earlier. Thus the problem of estimating the domain of attraction, given a Lyapunov function can be reformulated as follows:

\[
\text{max } \gamma \\
\text{subject to}
(V - \gamma)x^T x + r(x) \Delta V \geq 0 \tag{10}
\]

While this approach will find accurate estimates of the domains of attraction for a given Lyapunov function, it does not optimize the Lyapunov function to maximize the domain of attraction. In the next section, an expanding interior algorithm, originally proposed by [24], is introduced for the type of discrete time systems present in decentralized design processes.

### 3.2 Expanding Interior Algorithm for Discrete Time

The expanding interior algorithm maximizes level sets of a predefined closed curve such as a circle which is referred to as a shape function, instead of directly maximizing level sets of the Lyapunov function \( V \). The time derivative of the Lyapunov function, \( \Delta V \), is constrained to be negative definite within the region \( V = 1 \) and by maximizing the level sets of the shape function that can be contained within this region, \( V = 1 \), the algorithm indirectly maximizes the area of the curve \( V = 1 \).

When \( V \) is positive definite,

\[
\Omega = \{ x \in \mathbb{R}^n | V(k) \leq 1 \} \tag{11}
\]

is bounded and when

\[
\{ x \in \mathbb{R}^n | V(k) \leq 1 \} \subseteq \{ x \in \mathbb{R}^n | \Delta V \leq 0 \} \tag{12}
\]

\( \Omega \) is a subset of the domain of attraction for the system. To enlarge \( \Omega \), a variable sized region \( \Omega_p \subseteq \Omega \) is defined as follows:

\[
\Omega_p = \{ x \in \mathbb{R}^n | p(x(k)) \leq \beta \} \tag{13}
\]

where \( p(x(k)) \) is a positive definite polynomial called the shape function. This shape function \( p(x(k)) = \beta \) is always constrained to lie within \( V = 1 \) and hence maximizing \( \beta \) results in maximizing the region contained within \( V = 1 \). An additional constraint that \( \Delta V \leq 0 \) in the region \( V \leq 1 \) ensures that the region \( V \leq 1 \) belongs to the domain of attraction. The problem of improving the domain of attraction can then be reformulated as an optimization problem

\[
\text{max } \beta \\
\text{subject to}
\{ x \in \mathbb{R}^n | V(k) \leq 0, x \neq 0 \} = \phi \tag{14}
\]

\[
\{ x \in \mathbb{R}^n | p(x(k)) \leq \beta, V(k) \geq 1, V(k) \neq 1 \} = \phi
\]

\[
\{ x \in \mathbb{R}^n | V(k) \leq 1, \Delta V(k) \geq 0, x \neq 0 \} = \phi
\]

where \( \phi \) represents the null set. To reformulate the problem in a way that it can be solved by SOS programming, the constraint \( x \neq 0 \) is replaced with \( l_i(x) \neq 0 \), where \( l_i(x) \neq 0 \) is an SOS. Thus \( l_i(x) = 0 \) if and only if \( x = 0 \). Applying the Positivstellensatz
theorem [27] the problem becomes

\[
\begin{align*}
\max_{V \in \mathbb{R}^n, V(0) = 0, k_1, k_2, s_i \in \mathbb{Z}_+} & \quad \beta \\
\text{subject to} & & s_1 - V s_2 + I_1^{2k_1} = 0 \quad (15) \\
& & s_3 + (\beta - p)s_4 + (V - 1)s_5 \\
& & + (\beta - p)(V - 1)s_6 + (V - 1)^2s_7 = 0 \\
& & s_7 + (1 - V)s_8 + \Delta V s_9 + (1 - V)\Delta V s_{10} + (I_2)^2s_{13} = 0
\end{align*}
\]

where \( \Sigma_n \) is the set of all sum of squares polynomials in \( n \) variables. Thus the \( s_i \) and \( l_i \) terms are sum of squares polynomials. The problem can be reduced to a much simpler form by picking convenient values for some of the \( k_i \) and \( s_i \) terms. Picking \( k_1 = k_2 = k_3 = 1, s_2 = l_1, s_3 = s_4 = s_{10} = 0 \) and simplifying we get,

\[
\begin{align*}
\max_{V \in \mathbb{R}^n, V(0) = 0, s_i, s_j, s_9 \in \Sigma_n} & \quad \beta \\
\text{subject to} & & V - l_1 = s_1 \quad (16) \\
& & -((\beta - p)s_6 + (V - 1)) = s_5 \\
& & -((1 - V)s_8 + \Delta V s_9 + l_2) = s_7
\end{align*}
\]

Or alternatively the problem becomes

\[
\begin{align*}
\max_{V \in \mathbb{R}^n, V(0) = 0, s_i, s_j, s_9 \in \Sigma_n} & \quad \beta \\
\text{subject to} & & V - l_1 \in \Sigma_n \quad (17) \\
& & -((\beta - p)s_6 + (V - 1)) \in \Sigma_n \\
& & -((1 - V)s_8 + \Delta V s_9 + l_2) \in \Sigma_n
\end{align*}
\]

Since the above formulation has products of unknowns, it cannot be solved directly by SOS programming using SOS-TOOLS [28] (a MATLAB toolbox for solving sum of squares programs). An iterative approach to solving the above problem is as follows:

**Algorithm**

The following algorithm is a modification of the approach presented by Jarvis-Włoszek [24]. The problem formulation by Jarvis-Włoszek [24] is for continuous time systems. The algorithm presented here is modified for discrete time systems.

Let \( i \) be the iteration index. Pick the maximum degree of the Lyapunov function, the SOS multipliers and the \( l \) polynomials to be \( d_V, d_{s_i}, d_{s_j}, d_{s_9}, d_{l_1}, d_{l_2} \) respectively where \( d_V \) is the degree of polynomial \( V \).

**Initialization:** Denote the initial Lyapunov function \( V(i=0) \). Set \( \beta(i=0) = 0 \) and \( i = 1 \).

1. **Step 1.** Set \( V = V^{i-1} \) and solve the linesearch on \( \beta \)

\[
\begin{align*}
\max_{V \in \mathbb{R}^n, V(0) = 0, s_i, s_j, s_9 \in \Sigma_n} & \quad J = \beta \\
\text{subject to} & & -((\beta - p)s_6 + (V - 1)) \in \Sigma_n \\
& & -((1 - V)s_8 + \Delta V s_9 + l_2) \in \Sigma_n
\end{align*}
\]

Set \( s_8(i) = s_9 \) and \( s_9(i) = s_9 \). Continue to step 2.

2. **Step 2.** Set \( s_8 = s_8^{(i)} \) and \( s_9 = s_9^{(i)} \) and solve the linesearch on \( \beta \)

\[
\begin{align*}
\max_{V \in \mathbb{R}^n, V(0) = 0, s_i, s_j, s_9 \in \Sigma_n} & \quad J = \beta \\
\text{subject to} & & V - l_1 \in \Sigma_n \\
& & -((\beta - p)s_6 + (V - 1)) \in \Sigma_n \\
& & -((1 - V)s_8 + \Delta V s_9 + l_2) \in \Sigma_n
\end{align*}
\]

Set \( \beta(i) = \beta \) and \( V^{(i)} = V \). If \( \beta(i) - \beta(i-1) \) is less than a specified tolerance go to step 3, else increment \( i \) and go to step 1. Here, since \( V \) and \( \Delta V \) are unknown \( s_6 \) and \( s_9 \) are fixed so that the unknown variables appear linearly in the constraints.

3. **Step 3.** The set \( \{ x \in \mathbb{R}^n | V^{(i)}(x) \leq 1 \} \) contains \( p(x) = \beta \) and is the largest estimate of the fixed points region of attraction.

This algorithm is implemented in the next section on a two subsystem design problem.

4 EXAMPLE

In this section, the benchmark thin-walled pressure vessel design with hemispherical ends, shown in Figure 1, is considered to illustrate the two approaches that have been presented in section 3. The nomenclature for this case study is taken from [23] and is presented in Table 2. While this is certainly not a complex problem necessary of decomposition and decentralization, it does provide an effective example to exercise and illustrate the developments of this work. Current work focuses on more complex problems that would necessitate decomposition.

The multi-objective problem is to minimize the weight and maximize the volume of the cylinder, subject to geometric and stress constraints. The internal pressure \( P \) and the material of the vessel are specified. The problem is assumed to involve two interacting design teams, one who controls the volume by manipulating the radius \( R \) and the length \( L \), and the second who controls
Table 2. NOMENCLATURE OF THE PRESSURE VESSEL

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>W</td>
<td>Weight of the pressure vessel</td>
<td>lbs</td>
</tr>
<tr>
<td>V</td>
<td>Volume</td>
<td>in³</td>
</tr>
<tr>
<td>R</td>
<td>Radius</td>
<td>in</td>
</tr>
<tr>
<td>T</td>
<td>Thickness</td>
<td>in</td>
</tr>
<tr>
<td>L</td>
<td>Length</td>
<td>in</td>
</tr>
<tr>
<td>P</td>
<td>Pressure inside the cylinder</td>
<td>klb</td>
</tr>
<tr>
<td>S_t</td>
<td>Material allowable tensile strength</td>
<td>klb</td>
</tr>
<tr>
<td>ρ</td>
<td>Density of the material</td>
<td>lbs/in³</td>
</tr>
<tr>
<td>σ_circ</td>
<td>Circumferential stress</td>
<td>lbs/in²</td>
</tr>
</tbody>
</table>

the weight via the variable, thickness $T$. The cost and constraints for the two teams are listed in Tables 3 and 4. The rational reaction sets of each subsystem can either be found analytically as in [17], or approximated as in [23]. The RRS approximations from [23] are shown below:

$$W = 2 + 1.75R + 0.2445R^2$$  \hspace{1cm} (18)

$$L = 85.45 - 34.45T + 20.10T^2$$  \hspace{1cm} (19)

The problem constants for these RRS are:

$$P = 3.89 \text{ klb}; \quad S_t = 35.0 \text{ klb}; \quad \rho = 0.283 \text{ lbs/in}^3$$  \hspace{1cm} (20)

The intersection of the two RRS surfaces corresponds to the equilibrium point:

$$R = 28.4 \text{ in}$$
$$L = 87.5 \text{ in}$$
$$T = 3.09 \text{ in}$$  \hspace{1cm} (21)

Subsystem VOL

Maximize

$$V(R, L) = \frac{4}{3} \pi R^3 + \pi R^2 L$$

Design Variables

$R$ and $L$

Stress constraint

$$\sigma_{circ} = \frac{PR}{T} \leq S_t$$

Geometric constraints

$$5T - R \leq 0$$
$$R + T - 40 \leq 0$$
$$L + 2R + 2T - 150 \leq 0$$

Side constraints

$$0.1 \leq R \leq 36$$
$$0.1 \leq L \leq 140$$

Table 3. MODEL OF SUBSYSTEM VOL

Normalizing the variables to lie between -1 and 1 and subsequently moving the equilibrium point to the origin results in the equations which represent the update equations for the iterative design process:

$$R(k+1) = 0.887 R(k) - 0.558 T(k)^2$$
$$L(k+1) = -0.525 T(k) + 0.287 T(k)^2$$
$$T(k+1) = 0.739 R(k) + 0.089 R(k)^2$$  \hspace{1cm} (22)

It can be seen that the $R$ and $T$ equations are functions of each other, implying that the update equation for $L$ does not play a role in the convergence and can be ignored for the determination of the domain of attraction. This is because none of the update equations depend upon $L$ and hence the dynamics of $L$ do not affect the system and can be ignored for stability considerations. Thus a system with only two state variables, $R$ and $T$ is considered for stability analysis.
Subsystem WGT

\[
\begin{align*}
\text{Minimize} & \quad W(R,T,L) = \rho \left[ \frac{2}{3} \pi (R + T)^3 + \pi (R + T)^2 L - \left( \frac{4}{3} \pi R^3 + \pi R^2 L \right) \right] \\
\text{Design Variables} & \quad T \\
\text{Stress constraint} & \quad \sigma_{\text{circ}} = \frac{PR}{T} \leq S_i \\
\text{Geometric constraints} & \quad 5T - R \leq 0 \\
& \quad R + T - 40 \leq 0 \\
& \quad L + 2R + 2T - 150 \leq 0 \\
\text{Side constraints} & \quad 0.5 \leq T \leq 6
\end{align*}
\]

<table>
<thead>
<tr>
<th>Table 4. MODEL OF SUBSYSTEM WGT</th>
</tr>
</thead>
<tbody>
<tr>
<td>The final update equations for the analysis are given by</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
R(k+1) &= 0.887T(k) - 0.558T^2(k) \\
T(k+1) &= 0.739R(k) + 0.089R^2(k)
\end{align*}
\]

(23) \hspace{5cm} (24)

Chanron et al. [23] illustrate the stability of the origin (equilibrium point) by exploiting the sum of squares approach using SOSTOOLS [28] to determine a Lyapunov function which illustrates that the origin is locally stable. A contour plot of the expression \(V(k+1) - V(k)\) which should be negative definite for stability, was used to illustrate the domain of attraction. The Lyapunov function proposed in [23], found through trial and error is shown in Eq. (25).

\[
V = 2.54R^2 + 0.40RT + 3.96T^2
\]

(25)

The Zero Detection Algorithm was employed to determine the provable domain of attraction for this Lyapunov function and Figure 2 illustrates the resulting domain of attraction. It can be seen that the ellipse \(V = \gamma = 1.24\), is tangential to the contour curve which corresponds to \(V(k+1) - V(k)=0\). The square box corresponds to the domain of interest, i.e., the normalized design space defined by the design variable side constraints.

It is clear from Figure 2 that there are large regions of the design space which are not included in the domain of attraction. To improve the estimate of the domain of attraction, the expanding-interior approach is studied.

For this example, an arbitrary shape function \(p(R,T) = R^2 + T^2\) is used. A sixth order Lyapunov function shown in Figure 3 is found using the expanding interior algorithm. It can be seen that this Lyapunov function proves stability in the entire design space \(-1.5766 \leq R \leq 0.4234, -0.9418 \leq T \leq 1.0582\). The convergence of the algorithm to a Lyapunov function depends on the choice of the shape function. That is, a different shape function will result in a different Lyapunov function.

In Figure 4, the new domain of attraction (with \(n_v = 6\)) is compared to the domain found in [23] (with \(n_v = 2\)). It is clear that the new domain of attraction is significantly better than the previous one.

The algorithm iteratively searches for a new Lyapunov func-
Design Region

\( n_v = 6 \)

\( n_v = 2 \)

Figure 4. COMPARING THE ESTIMATES OF THE DOMAIN OF ATTRACTION

Design Region

\( n_v = 6 \)

\( n_v = 2 \)

Figure 5. COMPARING THE DOMAIN OF ATTRACTION FOR DIFFERENT INITIAL LYAPUNOV FUNCTIONS

It is clear from Figure 5 that the initial Lyapunov function plays an important role in the determination of the domain of attraction of the equilibrium point. But, regardless, the provable domain of attraction using the two approaches is guaranteed to always be larger than the domains developed in [23].

5 CONCLUSIONS AND FUTURE WORK

Two approaches which exploit the Positivstellensatz theorem to determine the domain of attraction for continuous time systems have been modified to cater to discrete time systems. These approaches have been used to determine the stability region of equilibrium points of a decentralized design system. The benchmark pressure vessel design problem has been studied and the benefit of using the expanding interior approach over the zero detection approach has been demonstrated. The proposed approach can be automated to generate the domain of attraction for

\[
V = \begin{bmatrix} R \\ T \end{bmatrix} P \begin{bmatrix} R \\ T \end{bmatrix} = 3.1329R^2 + 2.7109T^2. \tag{30}
\]

The expanding interior algorithm is used to determine a sixth order Lyapunov function using Eq. (30) as the initial Lyapunov function. The domain of attraction for the resulting sixth order Lyapunov function (\( V_2 \)) is illustrated by the dashed line in Figure 5 and is compared to the Lyapunov function (\( V_1 \)) generated by initializing the algorithm with Eq. (25).

\[
\begin{align*}
R(k + 1) &= 0.887T(k) \\
T(k + 1) &= 0.739R(k).
\end{align*}
\]
generic discrete time systems. The expanding interior approach is an iterative algorithm which is not guaranteed to converge to a global optimum. This requires initiating the algorithm with different Lyapunov functions. The shape function is also instrumental in the effectiveness of the algorithm. But the two approaches presented, significantly improve the estimate of the domain of attraction. By generating improved domains of attraction, the convergence to a stable equilibrium solution of decentralized design problems can be guaranteed across a much wider range, and many times across the entire design space. This implies, that regardless of where the process starts, and which subsystem starts the process, it will always converge to a stable solution, acceptable to all the subsystems.

This work can be applied to current design processes in a number of effective ways:

- Large product design companies can utilize this approach to predict and understand potential outcomes of their design processes before having to invest large resources into a complex, iterative sequence of decisions. For instance, a large company with a complicated network of suppliers (e.g., Boeing) could use this work coupled with previous work in [22, 23] to not only predict possible design outcomes, but also the stability of the outcome. That is, a company could gain descriptive insight into its complex, decentralized network of decision makers and then use this to develop prescriptive insight into how the dynamics of the network need to be changed to produce an optimal and stable outcome. Changes could occur in process parameters (developing cooperative coalitions in the network) or in problem parameters (changing some decision models to avoid divergence or enhance stability).

- Most of the work in this paper can and has been automated. Therefore, engineers with an understanding of the underlying decision model can gain some understanding about process stability quickly and without having to learn the control theory mathematics. Some of the existing fundamental software tools from the California Institute of Technology [28] were used in this work.

A necessary assumption of this work is that the decision models are represented by associated mathematics in the form of a basic optimization formulation. While this limits the work to the stages of design marked by mathematical models, it also allows for the broad application across the design stages where mathematical representations are typically used (e.g., embodiment design, detailed design).

Also at the core of distributed processes, is the issue of distributed rationality where individual subproblems each operate in a rational way which may not all align. That is, one subproblem decision maker may make decisions that are rational for them but completely irrational for the other subproblems. This creates scenarios of conflict and compromise that typically lead to suboptimal solutions. These solutions, more commonly known as Nash solutions are equilibrium points, but are not guaranteed to be optimal for any subproblem involved. The practical impact of this is noted in a study on the huge inefficiencies of the distributed decisions that are made in the design and fabrication of a building when designers, architects, engineers, developers and builders each make decision that serve their own interests [29]. Therefore, in future work, we will focus on developing convergence characteristics, including domains of attractions that lead the process to collective optimal solutions (also known as Pareto solutions).

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REFERENCES


